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ONE-VELOCITY HETEROGENEOUS FLOW NEAR A CONE

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The self-similar flow of a one-velocity multicomponent mixture near a cone with an attached shock wave is studied within the framework of the generalized-equilibrium model of a heterogeneous medium. The flow generalizes the known Busemann solution for an ideal gas. Results of the numerical simulation of the problem of flow of a gas–liquid mixture past a cone are presented.

The one-velocity model of a disperse medium is used to describe shock-wave processes in foamy media, bubble liquids, water-saturated grounds, and other such systems. Both time- and space-averaged values of the parameters are used in the model (small-scale fluctuations are disregarded). This model of a disperse medium which was suggested in [1] for the first time, has been augmented and developed in [2]. In the present work, we study the self-similar flow of a disperse medium near a cone which generalizes the known Busemann solution for an ideal gas to multicomponent mixtures. Cone flows were not considered within the framework of the model used earlier. We note the studies [3, 4], where a close problem — flow of a gas–liquid mixture near a wedge — has been studied both theoretically and experimentally.

Formulation of the Problem. We consider the stationary flow of a multicomponent mixture with an attached conical shock wave past a cone at a zero angle of attack (Fig. 1). Resolving the velocity vectors before and after the shock into normal components and those tangential to the front of the attached shock wave, we obtain $u_{n0} = |\mathbf{u}_0| \sin\beta$, $u_{t0} = |\mathbf{u}_0| \cos\beta$, $u_{n.s} = |\mathbf{u}_s| \sin\delta$, and $u_{t.s} = |\mathbf{u}_s| \cos\delta$. The subscripts 0 and s denote the parameters of the mixture ahead of the front of the attached shock wave and on the front. Since the tangential components of the velocity are the same for both vectors $\mathbf{u}_0$ and $\mathbf{u}_s$, we have

$$|\mathbf{u}_0| \cos\beta = |\mathbf{u}_s| \cos\delta . \tag{1}$$

In contrast to the plane case (flow past a wedge), the velocity vector $\mathbf{u}_s$ is noncollinear to the generatrix of the cone surface. On passage through the attached shock wave, one should use the normal velocity components in the Rankine–Hugoniot relations for the normal shock; thus, we have

$$\rho_0 |\mathbf{u}_0| \sin\beta = \rho_s |\mathbf{u}_s| \sin\delta , \quad p_0 + \rho_0 |\mathbf{u}_0|^2 \sin^2\beta = p_s + \rho_s |\mathbf{u}_s|^2 \sin^2\delta , \tag{2}$$

where, for example, $\rho_0 = \sum_{i=1}^{n} \alpha_{i0}\rho_{i0}^0$ is the density of the undisturbed mixture $\left(\sum_{i=0}^{n} \alpha_{i0} = 1\right)$ and ρ_{i0} and α_{i0} are the true density and volume fraction of the ith component of the mixture. The relations presented above are considered in combination with the shock adiabat of the n-component mixture [2], in which the first m fractions are taken to be compressible:

$$\frac{\rho_0}{\rho} = \sum_{i=1}^{m} \alpha_{i0} \left(\frac{\chi_i (p + p_{\bullet i}) + p_0 + p_{\bullet i}}{\chi_i (p_0 + p_{\bullet i}) + p + p_{\bullet i}}\right) + \sum_{i=m+1}^{n} \alpha_{i0} , \tag{3}$$

where $\chi_i = (\gamma_{\bullet i} - 1)/(\gamma_{\bullet i} + 1)$ and $p_{\bullet i} = \rho_{\bullet i} c_{\bullet i}^2/\gamma_{\bullet i}$. Here $\gamma_{\bullet i}$, $\rho_{\bullet i}$, and $c_{\bullet i}$ are the constants of the two-termed equation of state

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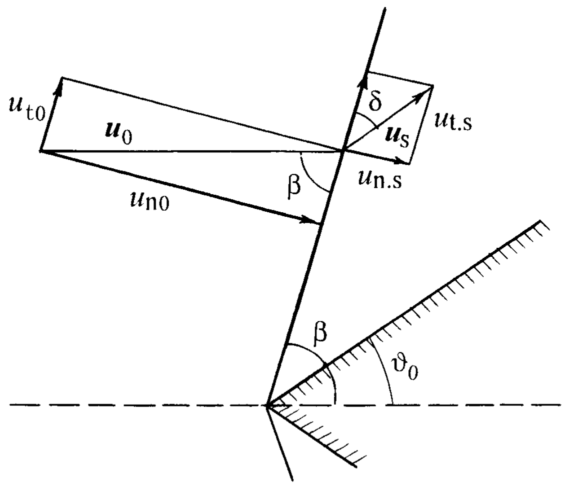


Fig. 1. Flow of a disperse medium near the cone.

$$\varepsilon_i = \frac{p - c_{\bullet i}^2 (\rho_i^0 - \rho_{\bullet i})}{(\gamma_{\bullet i} - 1)\, \rho_i^0},$$

which determine the properties of the ith compressible component. From (1) and (2) we have the following expressions:

$$|\mathbf{u}_s| = |\mathbf{u}_0| \frac{\cos\beta}{\cos\delta}, \quad \rho_s = \rho_0 \frac{\tan\beta}{\tan\delta}, \tag{4}$$

the substitution of which into the second relation of (2) yields

$$p_s = p_0 + \sin 2\beta\, (\tan\beta - \tan\delta) \frac{\rho_0 |\mathbf{u}_0|^2}{2}. \tag{5}$$

We consider the two-component mixture of an ideal gas with an incompressible condensed component in more detail. The shock adiabat for it has the form

$$\frac{\rho_0}{\rho_s} = \alpha_0 \frac{\chi p_s + p_0}{\chi p_0 + p_s} + 1 - \alpha_0; \quad \chi = \frac{\gamma - 1}{\gamma + 1}. \tag{6}$$

From (4)–(6) we have

$$\frac{\tan\delta}{\tan\beta} = \alpha_0 \frac{2p_0 (1 + \chi) + \rho_0 |\mathbf{u}_0|^2 \chi (\tan\beta - \tan\delta) \sin 2\beta}{2p_0 (1 + \chi) + \rho_0 |\mathbf{u}_0|^2 (\tan\beta - \tan\delta) \sin 2\beta} + 1 - \alpha_0. \tag{7}$$

Setting the angle β to be known, we determine δ, as follows from (7), from the relation

$$\delta = \arctan\left(\frac{B_2 \pm \sqrt{B_2^2 - 4B_1B_3}}{2B_1}\right), \tag{8}$$

where

$$B_1 = \frac{\rho_0 |\mathbf{u}_0|^2 \sin 2\beta}{2p_0}; \quad B_2 = 1 + \chi + B_1 (2 + \alpha_0 (\chi - 1)) \tan\beta; \quad B_3 = (B_1 (1 + \alpha_0 (\chi - 1)) \tan\beta + \chi + 1) \tan\beta.$$

The solution with a minus sign in front of the root in (8) has a physical meaning. The concentration of the gas behind the shock-wave front is calculated from the expression

$$\alpha_s = \frac{\alpha_0 \,(p_0 + \chi\, p_s)}{(1 + \alpha_0\,(\chi - 1))\, p_s + (\chi - \alpha_0\,(\chi - 1))\, p_0}\,.$$

The remaining parameters on the shock are calculated from (4) and (5). We note that in the general case of a multi-component mixture the angle δ cannot be expressed explicitly in terms of β and the parameters of the oncoming flow. To calculate this angle, we must solve the nonlinear equation

$$\frac{\tan\delta}{\tan\beta} = \sum_{i=1}^{m} \alpha_{i0}\, \frac{2\,(p_0 + p_{\bullet i})\,(\chi_i + 1) + \chi_i\,\rho_0\,|\mathbf{u}_0|^2\,(\tan\beta - \tan\delta)\sin 2\beta}{2\,(p_0 + p_{\bullet i})\,(\chi_i + 1) + \rho_0\,|\mathbf{u}_0|^2\,(\tan\beta - \tan\delta)\sin 2\beta} + \sum_{i=m+1}^{n} \alpha_{i0}\,.$$

Allowing for the fact that behind the front of the shock wave the flow is irrotational, we write the condition of the absence of vortices in a polar coordinate system (Fig. 1) in the form

$$\frac{du_r}{d\vartheta} = u_\vartheta\,. \tag{9}$$

The law of conservation of mass in the coordinate system used yields

$$\frac{du_\vartheta}{d\vartheta} = -\,2u_r - u_\vartheta\left(\operatorname{ctan}\vartheta + \frac{d\ln\rho}{d\vartheta}\right). \tag{10}$$

Equations (9) and (10) are obtained from the assumption that the parameters of the flow behind the shock are independent of the radius r (we seek the self-similar solution of the problem).

The Bernoulli integral for the nth component of the mixture has the form [5]

$$\rho_s\,\frac{u_r^2 + u_\vartheta^2 - |\mathbf{u}_s|^2}{2} = \sum_{i=1}^{m} \frac{\alpha_{is}\,\gamma_{\bullet i}\,(p_s + p_{\bullet i})}{\gamma_{\bullet i} - 1}\left[1 - \left(\frac{p_s + p_{\bullet i}}{p + p_{\bullet i}}\right)^{\frac{\gamma_{\bullet i} - 1}{\gamma_{\bullet i}}}\right] - (p - p_s)\sum_{i=m+1}^{n} \alpha_{is}\,. \tag{11}$$

In the partial case of a two-phase mixture of a gas with one incompressible component the Bernoulli integral takes on the form

$$\frac{\gamma\,(\gamma - 1)}{2\alpha_s c_s^2}\left(u_r^2 + u_\vartheta^2 - |\mathbf{u}_s|^2\right) + \frac{\gamma\rho_s - (1 - \alpha_s)\,\rho}{\rho}\left(\frac{\alpha_s\,\rho}{\rho_s - (1 - \alpha_s)\,\rho}\right)^{\gamma} = \alpha_s + \gamma - 1\,, \tag{12}$$

where $c_s = \sqrt{\gamma p_s / (\alpha_s \rho_s)}$ is the velocity of sound on the shock front. Differentiating (12) with respect to ϑ, we obtain

$$\frac{d\ln\rho}{d\vartheta} = -\,\frac{u_\vartheta\,\rho^2\left(u_\vartheta + \dfrac{du_\vartheta}{d\vartheta}\right)}{(\rho_s c_s)^2\left(\dfrac{\alpha_s\,\rho}{\rho_s - (1 - \alpha_s)\,\rho}\right)^{\gamma+1}}\,. \tag{13}$$

System (9)–(10) is rewritten with account for (13) in a form convenient for integration:

$$\frac{du_r}{d\vartheta} = u_\vartheta\,,$$

$$\frac{du_\vartheta}{d\vartheta} = \frac{\rho^2 u_\vartheta^3 - (\rho_s c_s)^2 (2u_r + u_\vartheta \operatorname{ctan} \vartheta)\left(\dfrac{\alpha_s \rho}{\rho_s - (1-\alpha_s)\rho}\right)^{\gamma+1}}{(\rho_s c_s)^2 \left(\dfrac{\alpha_s \rho}{\rho_s - (1-\alpha_s)\rho}\right)^{\gamma+1} - (\rho u_\vartheta)^2},$$

$$\frac{d\rho}{d\vartheta} = \frac{u_\vartheta \rho^3 (2u_r + u_\vartheta \operatorname{ctan} \vartheta - u_\vartheta)}{(\rho_s c_s)^2 \left(\dfrac{\alpha_s \rho}{\rho_s - (1-\alpha_s)\rho}\right)^{\gamma+1} - (\rho u_\vartheta)^2}. \tag{14}$$

A system of ordinary differential equations for a multicomponent mixture is written similarly; in this case the Bernoulli integral in general form (11) must be used instead of (12). The corresponding system of equations has the form

$$\frac{du_r}{d\vartheta} = u_\vartheta, \quad \frac{du_\vartheta}{d\vartheta} = -\frac{(2u_r + u_\vartheta \operatorname{ctan} \vartheta) f_1(p) + \rho u_\vartheta^2 u_r f_2(p)}{f_1(p) + \rho u_\vartheta^2 f_2(p)},$$

$$\frac{d\rho}{d\vartheta} = -\frac{u_\vartheta \rho^2 (u_r + u_\vartheta \operatorname{ctan} \vartheta) f_2(p)}{f_1(p) + \rho u_\vartheta^2 f_2(p)}, \quad \frac{dp}{d\vartheta} = -\frac{\rho_s u_\vartheta (u_r + u_\vartheta \operatorname{ctan} \vartheta)}{f_1(p) + \rho u_\vartheta^2 f_2(p)}, \tag{15}$$

where

$$f_1(p) = \sum_{i=1}^{m} \alpha_{is} \left(\frac{p_s + p_{\bullet i}}{p + p_{\bullet i}}\right)^{\frac{2\gamma_{\bullet i} - 1}{\gamma_{\bullet i}}} - \sum_{i=m+1}^{n} \alpha_{is}; \quad f_2(p) = \sum_{i=1}^{m} \frac{\alpha_{is}}{\gamma_{\bullet i}(p + p_{\bullet i})} \left(\frac{p_s + p_{\bullet i}}{p + p_{\bullet i}}\right)^{1/\gamma_{\bullet i}}.$$

The solution of system (14) must, first, satisfy the boundary condition of nonflow through the cone surface, which yields $u_\vartheta = 0$ at $\vartheta = \vartheta_0$. Second, the Rankine–Hugoniot relations (1)–(5), which involve the angle of slope of the attached shock wave, must hold on the shock-wave front. It should be borne in mind that the angle β is *a priori* unknown; therefore, we replace the boundary-value problem with complex nonlinear conditions at the region boundaries by the Cauchy problem which makes the algorithm of calculation much more simple. To do this, we specify the angle β arbitrarily, then we determine the parameters of the flow on the shock-wave front (at $\vartheta = \beta$) from relations (4)–(8). Then, we integrate system (14) by a numerical method from the initial angle $\vartheta = \beta$ to such a value of it $\vartheta = \vartheta_0$ at which the nonflow condition is satisfied ($u_\vartheta = 0$). Thus, we determine the half-angle of the streamlined cone ϑ_0. By varying β we find the dependence $\vartheta_0(\beta)$, inverting which we determine the sought dependence $\beta(\vartheta_0)$. After the calculation of u_r, u_ϑ, and ρ from (14), the pressure p and the volume concentration of the gas in the mixture α are determined from the relations

$$p = p_s \left(\frac{\alpha_s \rho}{\rho_s - (1-\alpha_s)\rho}\right)^{\gamma}, \quad \alpha = \frac{\alpha_s p_s^{1/\gamma}}{\alpha_s p_s^{1/\gamma} + (1-\alpha_s) p^{1/\gamma}}, \tag{16}$$

which follow from the condition of isentropy of the flow behind the shock-wave front [2].

In the general case of a multicomponent mixture, the volume concentrations of the fractions are calculated from the relations

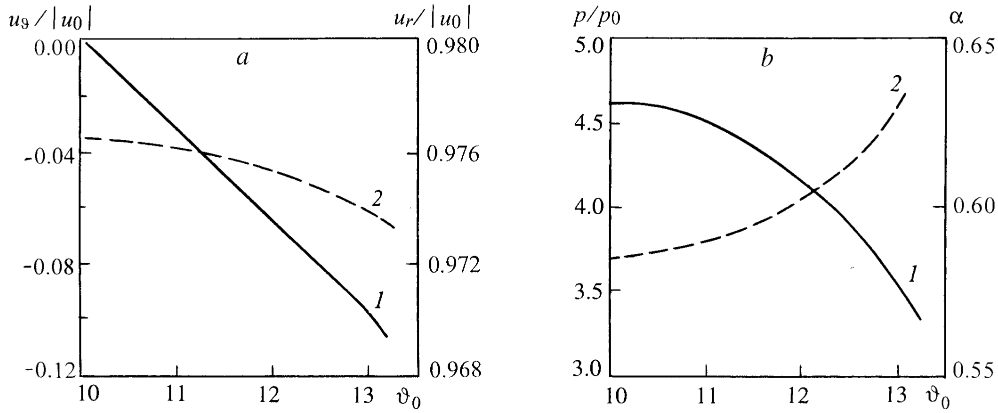


Fig. 2. Distribution of the parameters of the flow u_ϑ/u_0 (1), u_r/u_0 (2) [a] and α (1), p/p_0 (2) [b] behind the front of the attached shock wave as a function of ϑ_0 for different α_0.

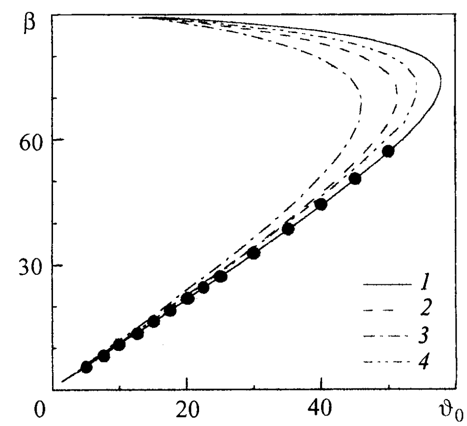


Fig. 3. Dependences of β on ϑ_0 for different α_0.

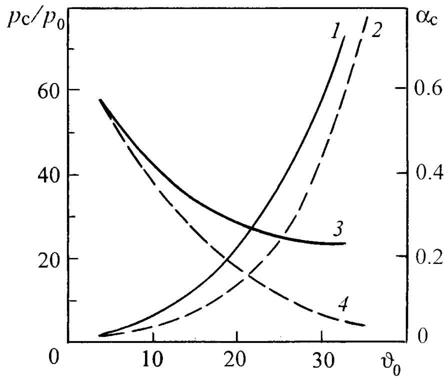
$$\alpha_i = \frac{\alpha_{is}\left(\dfrac{p_s + p_{\bullet i}}{p + p_{\bullet i}}\right)^{1/\gamma_{\bullet i}}}{\sum\limits_{i=1}^{m} \alpha_{is}\left(\dfrac{p_s + p_{\bullet i}}{p + p_{\bullet i}}\right)^{1/\gamma_{\bullet i}} + \sum\limits_{i=m+1}^{n} \alpha_{is}} .$$

Calculation Results. We consider the flow of a gas–liquid mixture with the volume concentration of the gas α_0 for the case of an incompressible liquid fraction and the parameters of the mixture p_0 = 0.1 MPa, γ = 1.4, ρ_{10}^0 = 1.19 kg/m^3, and ρ_{20}^0 = 1000 kg/m^3. We determine the Mach number in the oncoming flow from the relation

$$\mathrm{M} = |\mathbf{u}_0|/c_0 ,$$

where $c_0 = \sqrt{\gamma p_0/(\alpha_0\rho_0)}$ is the velocity of sound in an undisturbed mixture.

System (14) was solved numerically by the Runge–Kutta method. The distribution of the parameters of the flow behind the shock-wave front — the normalized components of velocity, pressure, and volume concentration of the gas in the mixture for M = 10, α_0 = 0.8, and $\vartheta_0 = 10^\mathrm{o}$ — is presented in Fig. 2 as a function of the angle ϑ. It is seen from the figure that at $\vartheta = \vartheta_0$ the velocity component u_ϑ decreases according to a nearly linear law from the initial value on the shock-wave front to zero. The pressure p and the radial velocity u_r increase monotonically with increase in ϑ, thus reaching their maximum values on the cone surface, whereas the concentration of the gas in the mixture decreases. We note that in the case of flow past a wedge the values of the parameters behind the front of the attached shock wave are constant and do not depend on the angle ϑ.



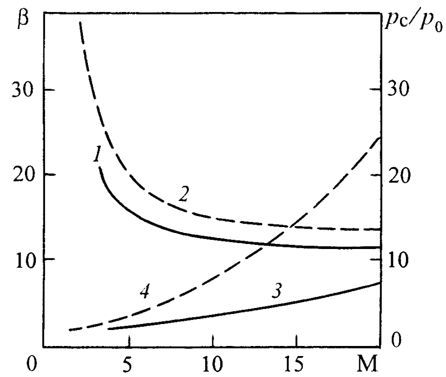


Fig. 4. Dependences of the parameters of the flow near the cone on ϑ_0 according to the adiabatic (1, 3) and isothermal (2, 4) models.

Fig. 5. Dependences of the parameters of the flow on the Mach number for a cone (1, 3) and a wedge (2, 4).

Figure 3 presents the dependences of the angle of the attached shock wave β on ϑ which are calculated for M = 40 and a concentration of gas in the mixture of α_0 = 1, 0.9, and 0.8 (curves 1–3). Dependence 4 is obtained within the framework of the isothermal model (see below) for $\alpha_0 = 0.8$ and M = 40. We note that the calculated data for a pure gas ($\alpha_0 = 1$) coincide with the corresponding tabular data from, for example, [6] (dots in Fig. 3). It is seen from Fig. 3 that two angles β, which determine the positions of weak and strong attached shock waves, exist for each angle ϑ_0. It is not necessary to consider the strong attached shock wave corresponding to a higher value of β [7]. The smaller the volume concentration of the gas in the mixture, the larger the angle of deviation of the attached shock wave, which is a consequence of the nonlinear dependence of the properties of the mixture on the concentration of the gas. Just as in the plane case [3], for the given Mach number and concentration of the gas in the mixture we have a limit angle ϑ_{0*}; when this angle is exceeded the mode of flow with an attached shock wave is impossible, since a separated shock wave is formed. The critical angle ϑ_{0*} decreases with decrease in the concentration of gas in the mixture (Fig. 3). We note that in contrast to the cone, for a wedge it is possible to find a limit angle corresponding to M = ∞ [3].

It is well known that in gas–liquid systems with a low concentration of gas in the mixture one must use, in a number of cases, the isothermal model of a disperse medium. Formally, transition from the adiabatic model to an isothermal one is reached if the adiabatic exponent of the gas on the shock-wave front is set to be equal to unity [3]. Figure 4 shows the dependences $(p_c/p_0)(\vartheta)$ (curves 1, 2) and $\alpha_c(\vartheta)$ (3, 4) calculated according to the adiabatic (solid lines) and isothermal (dashed lines) models in the case of flow of a gas–liquid mixture with Mach number M = 10 and the concentration of gas in an undisturbed flow $\alpha_0 = 0.7$ past a cone.

Figure 5 presents the dependences of the angle of slope β of the attached shock wave (1, 2) and the pressure ratios p_c/p_0 (3, 4) on the Mach number for a cone (solid curves) and a wedge (dashed curves) with an angle of $\vartheta_0 = 10^{\circ}$ and a concentration of gas in the mixture of $\alpha_0 = 0.95$ ($\gamma = 1.4$) calculated within the framework of the adiabatic model. It is seen from Fig. 5 that in the case of the cone both the angles of slope of the attached shock wave and the values of the pressure near the obstacle are lower than the corresponding values for the wedge. The divergences of pressure increase with increase in the Mach number. The behavior of the dependences is similar for the case of an ideal gas [6].

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NOTATION

(r, ϑ), axes of the polar coordinate system; ϑ_0, half-angle of the cone, deg; β, angle of slope of the attached shock wave, deg; δ, angle between the generatrix of the attached shock wave and the velocity vector of the particles of the medium behind the shock-wave front, deg; p, pressure, Pa; ε, specific internal energy; u, velocity, m/sec; γ, adiabatic exponent of the gas; c, velocity of sound, m/sec; ρ, mixture density, kg/m^3; ρ_i^0, true density of the ith fraction, kg/m^3; α_i, volume concentration of the ith component of the mixture. Subscripts: 0, oncoming (undisturbed) flow; s, on the shock front; c, on the cone surface; n and t, normal and tangential components; *, critical value; •, for the constants of the two-termed equation of state.

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